Abstract

These notes will provide a rough translation of the account of bundles of principal parts as set out in SGA VI, II Appendice II [1]. Some notation has been modernized, but this translation attempts to remain faithful to the original text. Any errors are undoubtedly that of the translator.

Mixed Structures

[1, II App. II, §1.1].

Fix a topological space S. Let $V \subseteq \mathbb{R}^n$ open. For $r \in \mathbb{N}$, we denote $C^r(V)$ the space of real-valued Frechet functions of type C^r on V. We let $C^\infty(V)$ be the limit $\lim_{\leftarrow r} C^r(V)$. Let $U \subseteq S$ be open. For $r \in \mathbb{N} \cup \{\infty\}$, we notice the sheaf ${}^r\mathcal{O}_{U\times V}$ the sheaf of rings on $U \times V$ defined by

$$\Gamma(U' \times V',^r \mathcal{O}_{U\times V}) = C(U', C^r(V))$$

where U' (resp. V') are open sets in U (resp. V), and $C(U', C^r(V'))$ denotes the space of continuous maps from U' to $C^r(V')$. There is a canonical injection

$$\mathrm{pr}_1^{-1}(\mathcal{O}_U) \to^r \mathcal{O}_{U\times V},$$

which makes $(U \times V,^r \mathcal{O}_{U\times V})$ a ringed space over U.

We say that a ringed space X over S is a *mixed variety of class* C^r ($r \in \mathbb{N} \cup \{\infty\}$) if X is locally isomorphic (as a ringed space over S) to a space of type $(U \times V,^r \mathcal{O}_{U\times V})$. (If S is reduced to a point, a variety of mixed class C^r over S is none other than a variety of class C^r in the usual sense).

Let $f : X \to S$ be a map of varieties of class C^r. For $s \in S$, the fiber $X_s = f^{-1}(s)$ is equipped with the sheaf $i_s^{-1}(\mathcal{O}_X)/I_s$, where $i_s : X_s \to X$ is the injection from I_s the ideal of sections that vanish on X_s as a variety of class C^r.

Relative Differential Operators

[1, II App. II, §1.2].

Following EGA VI 16 and SGA 2 VII, an infinitesmal study of mixed varieties. We will confine ourselves to a quick review of the notions we will need.

[1, II App. II, §1.2.1].

Let $f : X \to S$ be a mixed variety of class C^∞. For $n \in \mathbb{N}$, define the sheaf $\mathcal{P}^n_{X/X}$ the *sheaf of principal parts of order* n *on* X. There are two projections pr_1 and pr_2 of $X \times_S X$ to X of $\mathcal{O}_X$-algebras over $\mathcal{O}_X$ locally free of finite type as an $\mathcal{O}_X$-module. When we do not specifiy the structure of an $\mathcal{O}_X$-module on $\mathcal{P}^n_{X/S}$, it will be understood that this is the one from pr_1. There is a canonical isomorphism

$$\mathcal{P}^0_{X/S} \cong \mathcal{O}_X$$

$$\mathcal{P}^1_{X/S} \cong \mathcal{O}_X \oplus \Omega^1_{X/S}$$

where $\Omega^1_{X/S}$ is the sheaf of relative differentials on X over S the *relative cotangent sheaf.* Notice $T_{X/S}$ is the dual of $\Omega^1_{X/S}$ which we denote the *relative tangent sheaf.* The sheaf $T_{X/S}$ (resp. $\Omega^1_{X/S}$) is the sheaf of sections of a finite rank vector bundle on X which we denote the *relative (co)tangent fiber* which is a mixed S-variety over X.

There is a $f^{-1}\mathcal{O}_S$-linear homomorphism (corrresponding to the structure of the $\mathcal{O}_X$-algebra defined by pr_2)

$$d^n_{X/S} : \mathcal{O}_X \to \mathcal{P}^n_{X/S}$$

which has a section of $\mathcal{O}_X$ associated to the principal part to order n, which we denote the *universal differential operator of order* n on X relative to S.

[1, II App. II, §1.2.2]

More generally, if $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of finite type, one can consider the sheaf $\mathcal{P}^n_{X/S}(\mathcal{E}) = \mathcal{P}^n_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}$[1] of principal parts of order n on $\mathcal{E}$ (the tesor product being defined by the second structure of an $\mathcal{O}_X$-module on $\mathcal{P}^n_{X/S}$), and we have a universal differential operator

$$d^n_{X/S}(\mathcal{E}) : \mathcal{E} \to \mathcal{P}^m_{X/S}(\mathcal{E}).$$

If $\mathcal{E}, \mathcal{F}$ are locally free $\mathcal{O}_X$-modules of finite type. By definition, a differential operator from $\mathcal{E}$ to $\mathcal{F}$ of order $\leq n$ is an $f^{-1}\mathcal{O}_X$-linear[2] homomorphism from $\mathcal{E}$ to $\mathcal{F}$ that factors through $d^n_{X/S}(\mathcal{E})$ into an $\mathcal{O}_X$-linear homomorhpism from $\mathcal{P}^n_{X/S}(\mathcal{E})$ to $\mathcal{F}$ (the factorization is unique). In other words, the operator $d^n_{X/S}(\mathcal{E})$ establishes a unique bijection

$$\mathrm{Hom}_{\mathcal{O}_X}\left(\mathcal{P}^n_{X/S}(\mathcal{E}), \mathcal{F}\right) \cong \mathrm{Diff}^n(\mathcal{E}, \mathcal{F}),$$

where $\mathrm{Diff}^n(\mathcal{E}, \mathcal{F})$ is the set of all differential operators $\mathcal{E}$ to $\mathcal{F}$ or order $\leq n$.

[1, II App. II, §1.2.3]

We will also have to consider differential operators with complex coefficients. Denote $(\mathcal{O}_S \otimes \mathbb{C})$ (resp. $(\mathcal{O}_X \otimes \mathbb{C})$) the complexification of $\mathcal{O}_S$ (resp. $\mathcal{O}_X$). Let $\mathcal{E}, \mathcal{F}$ be locally $(\mathcal{O}_X \otimes \mathbb{C})$-modules of finite type. By a differential operator $\mathcal{E}$ to $\mathcal{F}$ of order $\leq n$, we will mean a $f^{-1}(\mathcal{O}_S \otimes \mathbb{C})$-linear homomorphism $\mathcal{E}$ to $\mathcal{F}$ that factorizes through $d^n_{X/S}(\mathcal{E})$ as an $(\mathcal{O}_X \otimes \mathbb{C})$-linear homomorphism from $\mathcal{P}^n_{X/S}(\mathcal{E})$ to $\mathcal{F}$. If we denote $\mathrm{Diff}^n_{\mathbb{C}}(\mathcal{E}, \mathcal{F})$ this set of operators, the operator $d^n_{X/S}(\mathcal{E})$ defines a bijection

$$\mathrm{Hom}_{(\mathcal{O}_X \otimes \mathbb{C})}\left(\mathcal{P}^n_{X/S}(\mathcal{E}), \mathcal{F}\right) \cong \mathrm{Diff}^n_{\mathbb{C}}(\mathcal{E}, \mathcal{F}).$$

[1, II App. II, §1.2.4]

For $n \geq 1$, we have a canonical exact sequence

$$0 \longrightarrow \mathrm{Sym}^n(\Omega^1_{X/S}) \longrightarrow \mathcal{P}^n_{X/S} \longrightarrow \mathcal{P}^{n-1}_{X/S} \longrightarrow 0$$

From which, on $\mathcal{O}_X$-tensoring with a locally free $\mathcal{O}_X$-module (resp. $(\mathcal{O}_X \otimes \mathbb{C})$) of finite type, an exact sequence of $\mathcal{O}_X$ (resp. $(\mathcal{O}_X \otimes \mathbb{C})$-modules):

$$0 \longrightarrow \mathrm{Sym}^n(\Omega^1_{X/S}) \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \mathcal{P}^n_{X/S}(\mathcal{E}) \longrightarrow \mathcal{P}^{n-1}_{X/S}(\mathcal{E}) \longrightarrow 0.$$

For $\mathcal{F}$ an locally free $\mathcal{O}_X$ (resp. $(\mathcal{O}_X \otimes \mathbb{C})$-module) of finite type, one can apply the functor $\mathrm{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{F})$ (resp. $\mathrm{Hom}_{(\mathcal{O}_X \otimes \mathbb{C})}(\cdot, \mathcal{F})$) to the exact sequences above to deduce exact sequences

$$0 \longrightarrow \mathrm{Diff}^{n-1}(\mathcal{E}, \mathcal{F}) \longrightarrow \mathrm{Diff}^n(\mathcal{E}, \mathcal{F}) \xrightarrow{\sigma_n} \mathrm{Hom}_{\mathcal{O}_X}(\mathrm{Sym}^n(\Omega^1_{X/S}) \otimes_{\mathcal{O}_X} \mathcal{E}, \mathcal{F})$$

(resp.

$$0 \longrightarrow \mathrm{Diff}^{n-1}_{\mathbb{C}}(\mathcal{E}, \mathcal{F}) \longrightarrow \mathrm{Diff}^n_{\mathbb{C}}(\mathcal{E}, \mathcal{F}) \xrightarrow{\sigma_n} \mathrm{Hom}_{(\mathcal{O}_X \otimes \mathbb{C})}(\mathrm{Sym}^n(\Omega^1_{X/S}) \otimes_{\mathcal{O}_X} \mathcal{E}, \mathcal{F}))$$

If X is paracompact and flasque over $\mathcal{O}_X$[3], one can complete the sequence appending a zero on the right. The homomorphism σ_n is called a *symbol*; which associates a differential operator from $\mathcal{E}$ to $\mathcal{F}$ or order $\leq n$ a "homogeneous polynomial map of degree n from $\Omega^1_{X/S}$ to $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ (resp. $\mathrm{Hom}_{(\mathcal{O}_X \otimes \mathbb{C})}(\mathcal{E}, \mathcal{F})$)", which we can still consider as a homomorphism $p^x(\mathcal{E}) \to p^x(\mathcal{F})$ (where p is the projection on X to teh relative cotangent fiber), satisfying certain homogeneous conditions.

[1] The text discusses a tensor product but prints $\oplus$.
[2] Should this be $f^{-1}\mathcal{O}_S$-linear?
[3] This was confusing in the french, "si X est paracompact et $\mathcal{O}_X$-mou", taking "mou" to be "flasque".

Let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be locallly free $\mathcal{O}_X$-modules of finite type, $u \in \mathrm{Diff}^m(\mathcal{E}, \mathcal{F}), v \in \mathrm{Diff}^n(\mathcal{F}, \mathcal{G})$, from which $vu \in \mathrm{Diff}^{m+n}(\mathcal{E}, \mathcal{G})$. One can easily check the relation

$$\sigma_{m+n}(vu) = \sigma_n(v)\sigma_m(u).$$

(We hvae the same relationship for differential operators with complex coefficients.) In particular, if composing two differential operators is zero, then composing their symbols is as well.

References

[1] P. Berthelot, A. Grothendieck, and L. Illusie, eds. *Séminaire de géométrie algébrique du Bois Marie 1966/67, SGA 6.Dirigé par P. Berthelot, A. Grothendieck et L. Illusie, Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussilia, S. Kleiman, M. Raynaud et J. P. Serre. Théorie des intersections et théorème de Riemann-Roch.* French. Vol. 225. Lect. Notes Math. Springer, Cham, 1971.